Fundamental Groups: An Introduction

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1 Introduction

Broadly speaking, topology is concerned with the study of the geometric properties of spaces that do not depend upon precise measurements such as distance or angle, but rather a weaker notion of "nearness", a concept captured by open sets. In particular, topology studies which spaces are equivalent under, and which properties of spaces are preserved by, continuous deformations, such as *homeomorphisms*, under which notions such as distance are certainly not preserved. Properties that are preserved by homeomorphisms are called *topological invariants* [3, p. 119]. A key idea of topology is that, if two spaces have different topological invariants, then they cannot be related by a homeomorphism (i.e. they're not *homeomorphic*).

The fundamental group, as we'll show, is one of these topological invariants. It originates from a branch of topology called algebraic topology, an area of maths which involves associating algebraic structures, like groups, to topological spaces. None of these associated algebraic structures perfectly describe the topological space they derive from, but they capture some specific aspect(s) of it. In the case of the fundamental group, this aspect is the one-dimensional holes of a space.

Consider a disc in $\mathbb{R}^2$. In an intuitive sense, it has no holes. This means that any closed loop drawn in the disc can be "shrunk down" to a single point (see Figure 1a). Compare this to an annulus, also in $\mathbb{R}^2$. Clearly, the annulus has a single hole. This hole means that, while some loops can still be shrunk to a single point, any which "go around" the hole can't, as the hole gets in the way in the shrinking process (see Figure 1b). Intuitively speaking, it is these loops that form the fundamental group of a space. The group structure comes from the ability to trace two loops with the same "basepoint" in succession to form a new loop. Some care must be taken to define when two loops are considered equivalent, but this is the most general idea, which will be developed, made rigorous, and used throughout the essay. In Section 2, we will rigorously define these basics. In Section 3, we will use these basics to construct the fundamental group. In Section 4, we will perform some basic computations of the fundamental groups of specific spaces. By Section 5, we will be able to rigorously show that a disc is not homeomorphic to an annulus via their respective fundamental groups.

2 The basics

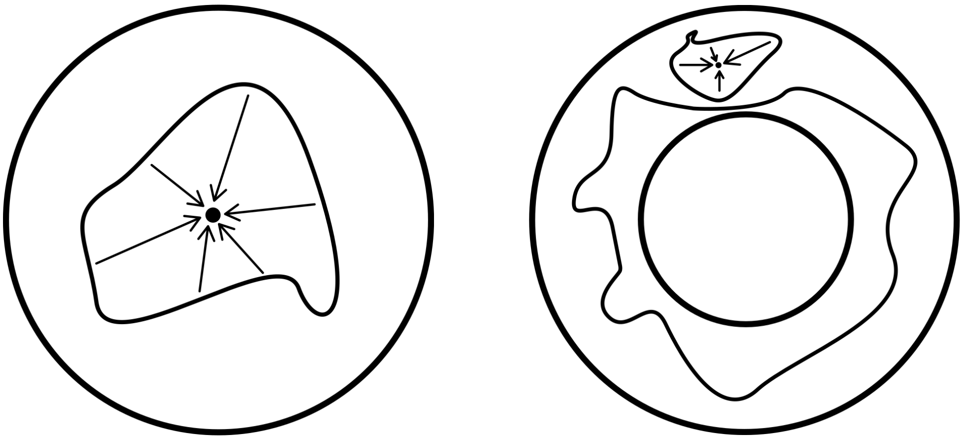
Before we can define the fundamental group, we must first build up some general theory, namely paths and loops, and homotopies.

2.1 Paths and loops

Our goal in defining the fundamental group is to capture the behaviour of loops and products of loops in topological spaces. In this section, we will aim to introduce the more general concept of paths, of which loops are a subset, as this increased generality will turn out to be useful later on.

We begin by introducing a simple bit of terminology.

[1]These figures are the author's own work.



(a) (b)

Figure 1: Loops on a disc can always be "shrunk down" to a point, but loops on an annulus can't always.[1]

Definition 2.1 (Map)**.** [1, p. 32] Let X and Y be topological spaces. A *map* from X to Y is a function from X to Y that is continuous.

Intuitively, a path refers to a specific "unit time" parametrisation of a curve in a space, such as the torus in Figure 2. However, to keep things general, the space does not have to be a metric space, but can be a more general topological space. This gives us the following definition.

Definition 2.2 (Path)**.** [4, p. 25] Let X be a topological space. A *path* in X is a map $f : [0,1] \to X$.

Note that, if f and g are paths in X with $f([0,1]) = g([0,1])$, we don't necessarily have $f = g$, as the specific parametrisation matters in the definition of a path. Moreover, for reasons that will become apparent, also note that we will typically use s as the generic input to a path, not t for "time" as might be expected.

Example 2.3. Consider the following simple paths in $\mathbb{R}^2$:
$f_1, f_2, f_3, f_4 : [0,1] \to \mathbb{R}^2$ such that

$$f_1(s) = (s,s), \qquad f_2(s) = (1-s, 1+2s),$$
$$f_3(s) = (4s^2, 4+\sin(2\pi s)), \qquad f_4(s) = (2,s).$$

These are illustrated in Figure 3.

[2]This figure is adapted by from an image by YassineMrabet [5], available under the CC BY-SA 3.0 license.

[3]This figure is the author's own work, produced using the Matplotlib Python library.

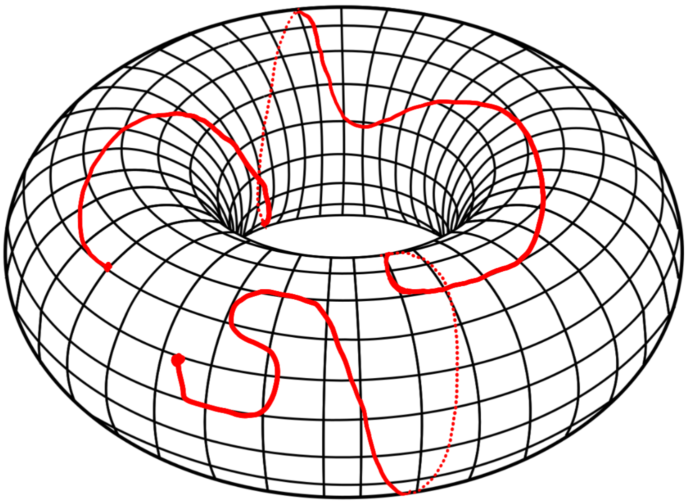


Figure 2: A path on a torus.[2]

Example 2.4. Consider the function $f : [0, 1] \to \mathbb{R}$ given by

$$f(s) = \begin{cases} s, & 0 \leq s \leq \frac{1}{2}, \\ 2s, & \frac{1}{2} < s \leq 1. \end{cases}$$

This is not a path as it is not continuous at $x = \frac{1}{2}$ and so not a map.

A constant path is also a path. It will be helpful to give this its own notation.

Definition 2.5 (Constant path)**.** [9, p. 47] Let X be a topological space and $x \in X$. The *constant path* at x is the path $\varepsilon_x : [0, 1] \to X$ given by $\varepsilon_x(s) = x$ for all $s \in [0, 1]$.

We can define a product of two paths f and g if f ends where g starts by following f and then following g, both at double speed in order to ensure that the resulting path is traversed in unit time.

Definition 2.6 (Multiplication of paths)**.** [4, p. 26] Let X be a topological space and let f and g be paths in X such that $f(1) = g(0)$. The *product* of f with g is a path, written $f \cdot g$, which first traverses f, then traverses g. That is,

$$(f \cdot g)(s) = \begin{cases} f(2s), & 0 \leq s \leq \frac{1}{2}, \\ g(2s - 1), & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Example 2.7. Consider f_1 and f_2 from Example 2.13. We can see that

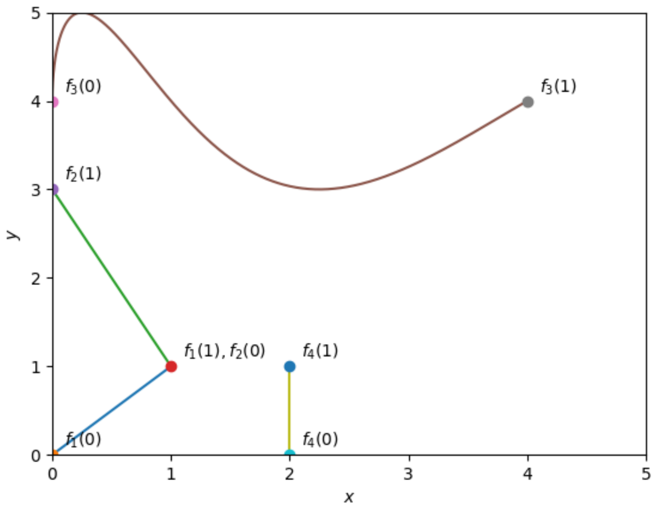


Figure 3: The paths f_1, f_2, f_3 and f_4 in $\mathbb{R}^2$.[3]

$f_1(1) = f_2(0)$. Hence, we may compute

$$(f_1 \cdot f_2)(s) = \begin{cases} f_1(2s) = (2s, 2s), & 0 \leq s \leq \frac{1}{2}, \\ f_2(2s-1) = (2-2s, 4s-1), & \frac{1}{2} \leq s \leq 1, \end{cases}$$

This product is illustrated in Figure 4.

Note that the product of two paths will always be continuous and hence a map, as is required of a path, by the following lemma.

Lemma 2.8 (Gluing lemma)**.** *[1, p. 69] Let Z and W be topological spaces, and let X, Y and $X \cup Y$ be subspaces of W with the subspace topology such that X and Y are closed in $X \cup Y$. Moreover, let $f : X \to Z$ and $g : Y \to Z$ be maps such that $f(x) = g(x)$ for all $x \in X \cap Y$. Then, the function $f \cup g : X \cup Y \to Z$ defined by $(f \cup g)(x) = f(x)$ for all $x \in X$ and $(f \cup g)(x) = g(x)$ for all $x \in Y$ is continuous.*

Proof. [1, p. 69] Let $C \subseteq Z$ be closed. Then, as f is continuous, $f^{-1}(C)$ is closed in X. Hence, as X is closed in $X \cup Y$, $f^{-1}(C)$ is closed in $X \cup Y$. Similarly, $g^{-1}(C)$ is closed in $X \cup Y$. Hence, $(f \cup g)^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$ is closed in $X \cup Y$ as the union of finitely many closed sets is closed. Hence, $f \cup g$ is continuous. $\square$

Hence, the product operation is clearly a closed operation. However, it is partial, as you can't always take the product of two arbitrary paths as the endpoints may not match up.

[4]This figure is the author's own work, produced using the Matplotlib Python library.

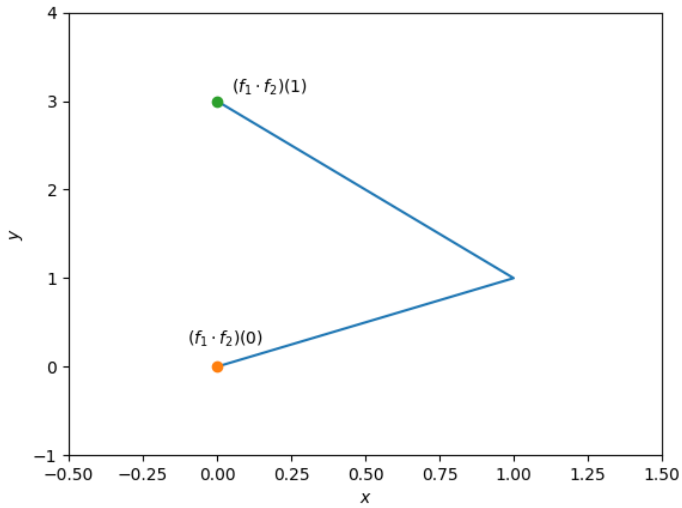


Figure 4: The product $f_1 \cdot f_2$ in $\mathbb{R}^2$.[4]

We will now define a loop, which is simply a path which starts and ends at the same point.

Definition 2.9 (Loop)**.** [4, p. 26] Let X be a topological space. A *loop* in X is a path α with $\alpha(0) = \alpha(1)$. The common value $\alpha(0) = \alpha(1) \in X$ is called the *basepoint*.

Remark 2.10. We will use Greek letters for loops and Latin letters for paths in general. Note that constant paths are loops, so fit this convention.

Example 2.11. Consider the following simple loops in $\mathbb{R}^2$:
$\alpha_1, \alpha_2, \alpha_3, \alpha_4 : [0, 1] \to \mathbb{R}^2$ such that

$$\begin{aligned}
\alpha_1(s) &= (\cos(2\pi s), \sin(2\pi s)),\\
\alpha_2(s) &= (3 + \cos(2\pi s) + \sin(2\pi s), 2 + \sin(2\pi s) + \sin(2\pi s)),\\
\alpha_3(s) &= \left(\frac{3}{2}(\cos(2\pi s))(1 - \cos(2\pi s)), \frac{3}{2}(\sin(2\pi s))(1 - \cos(2\pi s))\right),\\
\alpha_4(s) &= \varepsilon_{(-2,0)}(s) = (-2, 0).
\end{aligned}$$

These are illustrated in Figure 5.

Note that, when restricted to just loops with a fixed basepoint x, taking the product of two loops becomes a *binary operation.* Hence, it is natural to ask whether this gives rise to a group. After all, from the introduction,

[5]This figure is the author's own work, produced using the Matplotlib Python library.

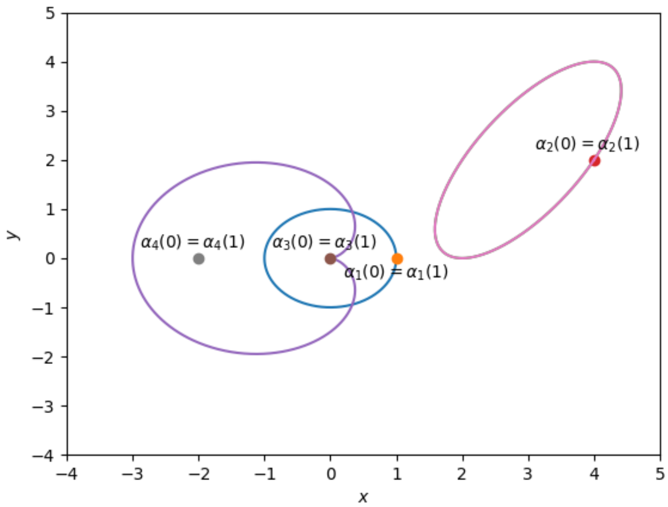


Figure 5: The loops $\alpha_1, \alpha_2, \alpha_3$ and α_4 in $\mathbb{R}^2$.[5]

we know that we are looking for a group built out of loops of some form. Unfortunately, we do not obtain a group. For example, we don't even have an identity element! It might seem that, for loops in a topological space X based at a point $p \in X$, ε_p acts as an identity. However, this is not the case. Consider a loop α in X based at p. Then, $(\alpha \cdot \varepsilon_p)(s) = \alpha(2s)$ for $0 \leq s \leq \frac{1}{2}$ and $(\alpha \cdot \varepsilon_p)(s) = p$ for $\frac{1}{2} \leq s \leq 1$, so $(\alpha \cdot \varepsilon_p)(s) \neq \alpha(s)$, the issue being that the product traverses the same curve, just twice as fast.

To solve this issue, we need to consider loops up to a concept called *homotopy*, which is introduced in full generality in the next section.

2.2 Homotopy

While we are most interested in homotopies of loops, we will define homotopy in full generality. We will see in later sections that this increased generality is useful.

Loosely speaking, a homotopy is a way to interpolate between two maps.

Definition 2.12 (Homotopy)**.** [1, p. 88] Let X and Y be topological spaces, and let $f, g : X \to Y$ be maps. Then, we say that f is *homotopic* to g if there exists a map $F : X \times [0,1] \to Y$ such that, for all $x \in X$, we have $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. When this is the case, we write $f \underset{F}{\simeq} g$. We call the function F a *homotopy*.

Sometimes, when the specific homotopy between f and g isn't important, we may simply write $f \simeq g$. In some books, such as Hatcher [4, p. 25], this notation is standard, with the homotopy specified separately when relevant.

We will now see some examples of homotopies.

Example 2.13. Let $f, g : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x$ and $g(x) = 2x$. Then, $f \underset{F}{\simeq} g$ where $F : \mathbb{R} \times [0,1] \to \mathbb{R}$ is given by $F(x,t) = x + tx$. Indeed, $F(x,0) = x + 0x = x = f(x)$, and $F(x,1) = x + 1x = 2x = g(x)$. This is illustrated in Figure 6.

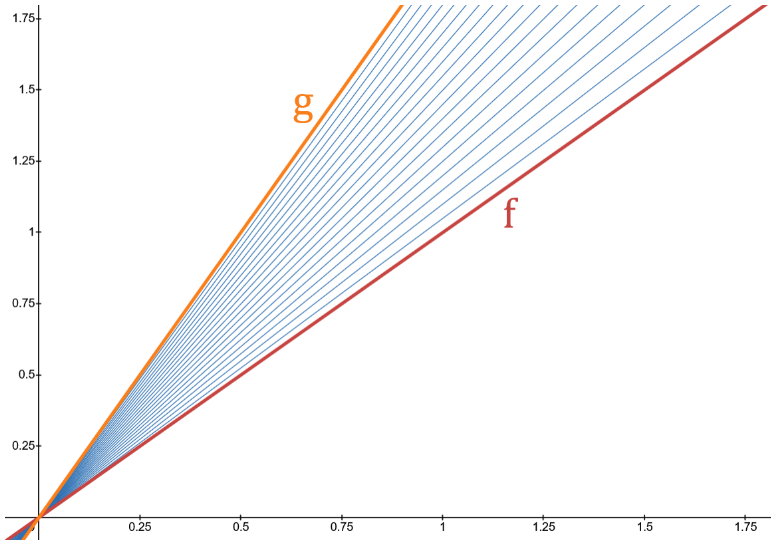


Figure 6: Example 2.13, with f in red, g in orange, and $F(\cdot, t)$ in blue for $t \in \{0.05, 0.1, \ldots, 0.95\}$.[6]

The example above is a specific case of the following important example of homotopy.

Example 2.14 (Straight-line homotopy)**.** [1, p. 88] Let X be a topological space, $C \subseteq \mathbb{R}^n$ be a convex subset, and $f, g : X \to C$ be arbitrary maps. Then, for any $x \in X$, $(1-t)f(x) + tg(x) \in C$ for all $x \in X$ and $t \in [0,1]$. Hence, $f \underset{F}{\simeq} g$, with $F(x,t) = (1-t)f(x) + tg(x)$ as, clearly, F is continuous and, for all $x \in X$, $F(x,0) = f(x)$ and $F(x,1) = g(x)$. We call F a *straight-line homotopy*.

In the following few examples, we will investigate homotopies of maps into S^n, which we must first define.

Definition 2.15 (Spherical n-space)**.** [8, p. 34] The unit sphere of $\mathbb{R}^{n+1}$ is called *spherical n-space*, denoted S^n. That is, $S^n := \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$.

Definition 2.16 (Antipodal points in S^n)**.** [8, p. 35] Two points $x, y \in S^n$ are *antipodal* if $x = -y$.

[6]This figure is the author's own work, primarily produced using the online graphing calculator Desmos.

Example 2.17 (Maps into S^n)**.** [1, p. 89] Let X be a topological space, and suppose that we have functions $f, g : X \to S^n$ such that $f(x)$ and $g(x)$ are never antipodal for any $x \in X$. Then, for a fixed $x \in X$, as $\mathbb{R}^{n+1}$ is convex, we can consider the line segment $(1-t)f(x)+tg(x)$, $t \in [0,1]$, connecting $f(x)$ and $g(x)$ in $\mathbb{R}^{n+1}$ (per Example 2.14). As $f(x)$ and $g(x)$ are not antipodal, they do not lie on the same diameter of S^n, so $(1-t)f(x)+tg(x) \neq 0$ for all $t \in [0,1]$ (see Figure 7). Hence, we may define $F : X \times [0,1] \to S^n$ by

$$F(x,t) = \frac{(1-t)f(x)+tg(x)}{\|(1-t)f(x)+tg(x)\|},$$

noting that

$$\|F(x,t)\| = \left\| \frac{(1-t)f(x)+tg(x)}{\|(1-t)f(x)+tg(x)\|} \right\| = \frac{\|(1-t)f(x)+tg(x)\|}{\|(1-t)f(x)+tg(x)\|} = 1,$$

so $F(x,t) \in S^1$ for all $x \in X$ and $t \in [0,1]$ as claimed. Observe that, for all $x \in X$, $F(x,0) = f(x)/\|f(x)\| = f(x)$ and, similarly, $F(x,1) = g(x)$. Moreover, F is clearly continuous. Therefore, we see that $f \underset{F}{\simeq} g$.

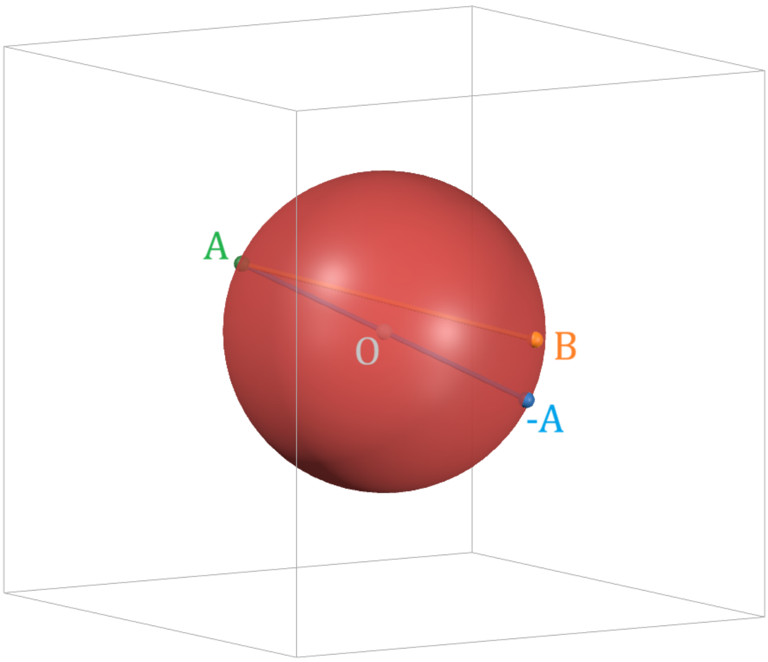


Figure 7: A (green) and $-A$ (blue) are antipodal, so the line connecting them (blue) passes through the origin (grey) and so takes the value 0. However, A and B (orange) are not antipodal, so the line connecting them (orange) does not pass through the origin. [7]

[7]This figure is the author's own work, primarily produced using the online 3D graphing calculator Desmos.

Definition 2.18 (Null homotopic)**.** [1, p. 91] Let X and Y be topological spaces. A map $f : X \to Y$ is called *null homotopic* if it is homotopic to a constant map.

Example 2.19 (More maps into S^n)**.** [8] Let X be a topological space and consider a non-surjective map $f : X \to S^n$. Then, f is null homotopic. To see this, note that, as f is not surjective, there must exist a point $y \in S^n$ such that $f(x) \neq y$ for all $x \in X$. Let the map $g : X \to S^n$ be given by $g(x) = -y$ for all $x \in X$. Then, $f(x) \neq -g(x)$ for all $x \in X$, so we see that $f \simeq g$ by Example 2.17.

We now introduce the useful concept of homotopy relative to a set.

Definition 2.20 (Homotopy relative to a set)**.** [1, p. 88] Let X and Y be topological spaces, and let A be a subset of X. Suppose that there exist maps $f, g : X \to Y$ with $f(x) = g(x)$ for all $x \in A$. Then, we say that f *is homotopic to* g *relative to* A if there exists a homotopy $F : X \times [0, 1] \to Y$ with $F(x, t) = f(x) = g(x)$ for all $x \in A$. In this case, we write $f \underset{F}{\simeq} g \operatorname{rel} A$.

Example 2.21. Building on Example 2.13, we can see that $f \underset{F}{\simeq} g \operatorname{rel} \{0\}$.

As the following example illustrates, two maps can be homotopic relative to different sets via different homotopies.

Example 2.22. [9] Let D be the unit disc in $\mathbb{R}^2$, that is $D = \left\{x \in \mathbb{R}^2 \,\middle|\, \|x\| \leq 1\right\}$, and let $f : D \to D$ be given by $f(r, \theta) = (r, \theta + 2\pi r)$ in polar coordinates. Moreover, let id_D be the identity map in D. Then, we have that $f \underset{F}{\simeq} \mathrm{id}_D \operatorname{rel} \{0\}$, with $F((r, \theta), t) = (r, \theta + 2\pi r(1 - t))$ in polar coordinates.

However, we also have that $f \underset{G}{\simeq} \mathrm{id}_D \operatorname{rel} S^1$, with $G = ((r, \theta), t) = (r, \theta + 2\pi(r-1)(1-t))$ in polar coordinates. To verify this, note that $G((r, \theta), 0) = (r, \theta + 2\pi(r - 1)(1 - 0)) = (r, \theta + 2\pi r - 2\pi) = (r, \theta + 2\pi r) = f(r, \theta)$, and $G((r, \theta), 1) = (r, \theta + 2\pi(r - 1)(1 - 1)) = (r, \theta) = \mathrm{id}_D(r, \theta)$ for all $(r, \theta) \in D$, and also, for all $(r, \theta) \in S^1$ and $t \in [0, 1]$, we have $G((r, \theta), t) = G((1, \theta), t) = (1, \theta + 2\pi(1 - 1)(1 - t)) = (1, \theta) = (r, \theta)$.

Homotopy is clearly a relation between maps. It is natural to ask if it forms an equivalence relation.

Proposition 2.23 (Homotopy is an equivalence relation)**.** *[1, p. 90] Given topological spaces X and Y, homotopy is an equivalence relation on the set of all maps from X to Y.*

Proof. See Appendix A. □

[8] This example is adapted from this author's solution to problem 5 of Armstrong §5 [1, p. 91].

[9] This example is partially adapted from this author's solution to problem 4 of Armstrong §5 [1, p. 91].

Corollary 2.24 (Relative homotopy is an equivalence relation)**.** *[1, p. 90] Given topological spaces X and Y and a subset A of X, homotopy relative to A is an equivalence relation on the set of all maps from X to Y which agree on A.*

Proof. [1, p. 90] The proof of Proposition 2.23 still holds with all homotopies relative to A. ☐

Before ending this section, we will show that homotopy interacts with function composition in the expected way.

Proposition 2.25 (Homotopy respects composition)**.** *[1, pp. 90–91] Let X, Y and Z be topological spaces with $A \subseteq X$ and $B \subseteq Y$.*

Suppose that we have maps $f, g : X \to Y$ and $h : Y \to Z$ such that $f \underset{F}{\simeq} g \operatorname{rel} A$ (Figure 8a). Then, $h \circ f \underset{h \circ F}{\simeq} h \circ g \operatorname{rel} A$.

Further, suppose instead that we have maps $f : X \to Y$ and $g, h : Y \to Z$ such that $g \underset{G}{\simeq} h \operatorname{rel} B$ (Figure 8b). Then, $g \circ f \underset{F}{\simeq} h \circ f \operatorname{rel} f^{-1}(B)$ with $F(x,t) = G(f(x),t)$.

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow F"] & Y \arrow[r, "h"] & Z \end{tikzcd}$$

(a)

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow G"] & Z \end{tikzcd}$$

(b)

Figure 8: The compositions and homotopies from the premises of Proposition 2.25, with homotopies represented by double arrows "⇒".[10]

Proof. See Appendix A. ☐

We now have the tools to construct the fundamental group, which we shall do in the next section.

3 The fundamental group

As discussed in the introduction, the idea behind the fundamental group is to somehow encode the holes of a topological space, and this is done by considering loops in this space. However, as we've already seen at the end of Section 2.1, loops under multiplication is not enough as there isn't even an identity loop. We need to instead consider loops up to homotopy.

[10]These figures based on figures from Armstrong [1, p. 91], produced using Quiver.

3.1 The construction

We will begin by defining the objects that will form the fundamental group.

Definition 3.1 (Homotopy classes of paths)**.** [11] Let X be a topological space and f a path in X. The *homotopy class* of f, denoted $[f]$, is the equivalence class of f under the equivalence relation of homotopy relative to $\{0, 1\}$ (see Corollary 2.24). That is, $[f] = \{g \text{ a path in } X \mid f \simeq g \operatorname{rel} \{0, 1\}\}$.

This can be intuitively thought of in a geometric sense, as in Figure 9, but this geometric intuition often doesn't hold in the most general topological spaces, so caution is required.

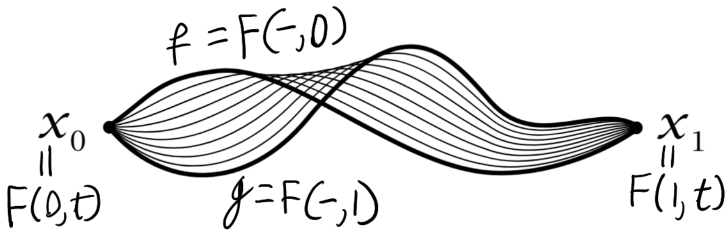


Figure 9: The homotopy $f \underset{F}{\simeq} g \operatorname{rel} \{0, 1\}$, represented visually.[12]

Homotopy classes are very well-behaved.

Proposition 3.2 (Products of paths respects homotopies)**.** *[4, p. 26] Let X be a topological space and let f_0, f_1, g_0 and g_1 be paths in X such that $f_0 \underset{F}{\simeq} f_1 \operatorname{rel} \{0, 1\}$, $g_0 \underset{G}{\simeq} g_1 \operatorname{rel} \{0, 1\}$, and $f_0(1) = f_1(1) = g_0(0) = g_1(0)$. Then, $f_0 \cdot g_0 \underset{H}{\simeq} f_1 \cdot g_1 \operatorname{rel} \{0, 1\}$, where $H(s, t) = F(2s, t)$ for $0 \le s \le \frac{1}{2}$ and $H(s, t) = G(2s - 1, t)$ for $\frac{1}{2} \le s \le 1$.*

Proof. See Appendix A. □

This lets us consider the product of the homotopy classes of two paths. That is, if X is a topological space and f and g are paths in X with $f(1) = g(0)$, then $[f] \cdot [g] := [f \cdot g]$, and this operation is well-defined by Proposition 3.2.

Theorem 3.3 (Homotopy classes of loops form a group)**.** *[1, p. 92] Let X be a topological space and let let $p \in X$. Then, the homotopy classes of loops in X based at p form a group with respect to multiplication of paths.*

Proof. [1, pp. 92–93] A sketch of the proof is given here. For a more detailed proof, see Appendix A.

[11] This definition comes from Armstrong [1, p. 92], but the notation is from Hatcher [4, p. 26].

[12] This figure is adapted from a figure from Hatcher [4, p. 25].

The fact that the product of homotopy classes of loops with a fixed basepoint is a binary operation follows immediately from the same fact for the product of loops themselves.

To show associativity, the existence of an identity, and the existence of inverses, it amounts to showing several equalities of homotopy classes. To show that, say, $[\alpha] = [\beta]$, the general strategy is to find a map f which is homotopic to the identity such that $\alpha = \beta \circ f$, and then to employ Proposition 2.25 to obtain that $\alpha \simeq \beta \operatorname{rel} \{0, 1\}$ and, hence, $[\alpha] = [\beta]$.

For associativity, for all loops α, β, γ based at p, we have $(\alpha \cdot \beta) \cdot \gamma = (\alpha \cdot (\beta \cdot \gamma)) \circ f$, where $f : [0, 1] \to [0, 1]$ is given by

$$f(s) = \begin{cases} 2s, & 0 \le s \le \frac{1}{4}, \\ s + \frac{1}{4}, & \frac{1}{4} \le s \le \frac{1}{2}, \\ \frac{s+1}{2}, & \frac{1}{2} \le s \le 1. \end{cases}$$

For identity, we see that $[\varepsilon_p]$ is the identity element (where ε_p is as per Definition 2.5). Then, for all α based at p, we have $\varepsilon_p \cdot \alpha = \alpha \circ f$, where $f : [0, 1] \to [0, 1]$ is given by

$$f(s) = \begin{cases} 0, & 0 \le s \le \frac{1}{2}, \\ 2s - 1, & \frac{1}{2} \le s \le 1. \end{cases}$$

We can similarly consider $\alpha \cdot \varepsilon_p = \alpha \circ g$ for some $g : [0, 1] \to [0, 1]$.

Finally, the inverse of $[\alpha] \in \pi_1(X, p)$ is $[\bar{\alpha}]$, where $\bar{\alpha}(s) = \alpha(1 - s)$ [4, p. 27]. We can show that this is well-defined, and then we have that $\alpha \cdot \bar{\alpha} = \alpha \circ f$, where $f : [0, 1] \to [0, 1]$ is given by

$$f(s) = \begin{cases} 2s, & 0 \le s \le \frac{1}{2}, \\ 2 - 2s, & \frac{1}{2} \le s \le 1. \end{cases}$$

We can similarly consider $\bar{\alpha} \cdot \alpha = \alpha \circ g$ for some $g : [0, 1] \to [0, 1]$. □

This group is precisely the fundamental group, which is formalised in the following definition.

Definition 3.4 (The fundamental group)**.** [1, p. 93] Let X be a topological space and $p \in X$. The group from Theorem 3.3 is called the *fundamental group* of X based at p, and is denoted $\pi_1(X, p)$. That is,

$$\pi_1(X, p) = \{[\alpha] \mid \alpha \text{ is loop in } X \text{ with basepoint } p\}$$

is a group under multiplication of paths.

The subscript "1" used in this notation immediately sticks out. We will return to what this signifies in the conclusion, Section 5.

Example 3.5 (Convex subsets of $\mathbb{R}^n$)**.** [4, p. 27] Let $C \subseteq \mathbb{R}^n$ be convex and let $p \in C$. Then, let $[\alpha], [\beta] \in \pi_1(C, p)$. Then, as per Example 2.14 with $X = [0, 1]$, we see that $\alpha \underset{F}{\simeq} \beta$ where F is the straight-line homotopy given by $F(s, t) = (1 - t)\alpha(s) + t\beta(s)$ (see Figure 10). In fact, we can see that, for all $t \in [0, 1]$, $F(0, t) = (1 - t)\alpha(0) + t\beta(0) = (1 - t)p + tp = p$ and, similarly, $F(1, t) = p$. Hence, we have that $\alpha \underset{F}{\simeq} \beta \operatorname{rel} \{0, 1\}$, so $[\alpha] = [\beta]$. In particular, for all $[\alpha] \in \pi_1(C, p)$, we have that $[\alpha] = [\varepsilon_p]$, the identity, so $\pi_1(C, p)$ is isomorphic to the trivial group.

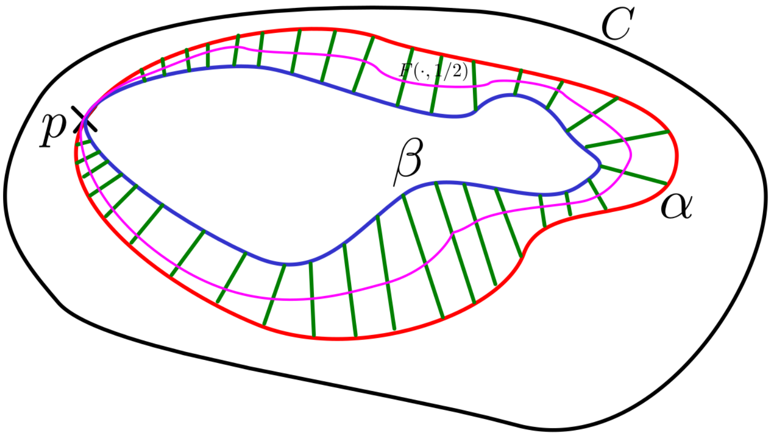


Figure 10: A visual representation of $\alpha \underset{F}{\simeq} \beta \operatorname{rel} \{0, 1\}$. The loops α and β, in red and blue respectively, are connected by an infinite number of straight lines, one for every pair of corresponding points (although only finitely many are shown here, in green). At each time t, the homotopy F forms a loop which cuts through these straight lines, with $F(\cdot, 1/2)$ shown here in magenta.[13]

So far, we've specified the basepoint when defining and computing the fundamental group. However, if we consider a space like $\mathbb{R}^2$, it should be clear that the choice of basepoint doesn't matter: all loops from any basepoint are homotopic to a constant loop from that basepoint, so the fundamental group is always trivial. In fact, as long as there is a path between two points in a space, then the fundamental groups of the space with those basepoints will be isomorphic: any loop at one basepoint, p, can be mapped to a loop at the other, q, simply by following a path from q to p, following the loop at p, and then following a path from p back to q. When considered up to homotopy, this is indeed an isomorphism, as we will show in the next theorem.

Theorem 3.6 (Basepoints don't matter)**.** *[1, p. 94] Let X be a path-connected topological space and $p, q \in X$. Then, $\pi_1(X, p) \cong \pi_1(X, q)$.*

[13] This figure is the author's own work.

Proof. [1, p. 94] As X is path-connected, there must exist a path f from p to q. Let $\bar{f}$ be the inverse path from q to p given by $\bar{f}(s) = f(1-s)$. Then, if α is a loop based at p, the path $(\bar{f} \cdot \alpha) \cdot f$ is a loop based at q (see Figure 11): $((\bar{f} \cdot \alpha) \cdot f)(0) = \bar{f}(0) = f(1-0) = f(1) = q$, and $((\bar{f} \cdot \alpha) \cdot f)(1) = f(1) = q$. Hence, we can define a function $\hat{f} : \pi_1(X, p) \to \pi_1(X, q)$ by $\hat{f}([\alpha]) = [\bar{f} \cdot \alpha \cdot f]$. This is well defined since, if $[\alpha] = [\beta]$, then $f([\alpha]) = [\bar{f} \cdot \alpha \cdot f] = [\bar{f}] \cdot [\alpha] \cdot [f] = [\bar{f}] \cdot [\beta] \cdot [f] = [\bar{f} \cdot \beta \cdot f] = f([\beta])$. Then, if $[\alpha], [\beta] \in \pi_1(X, p)$, we have that $\hat{f}([\alpha] \cdot [\beta]) = \hat{f}([\alpha \cdot \beta]) = [\bar{f} \cdot \alpha \cdot \beta \cdot f] = [\bar{f} \cdot \alpha \cdot f \cdot \bar{f} \cdot \beta \cdot f] = [\bar{f} \cdot \alpha \cdot f] \cdot [\bar{f} \cdot \beta \cdot f] = \hat{f}([\alpha]) \cdot \hat{f}([\beta])$, so $\hat{f}$ is a homomorphism. Moreover, $\hat{f}$ has an inverse $\hat{\bar{f}}$ given by $\hat{\bar{f}}([\alpha]) = [f \cdot \alpha \cdot \bar{f}]$. Hence, $\hat{f}$ is an isomorphism, so $\pi_1(X, p) \cong \pi_1(X, q)$. □

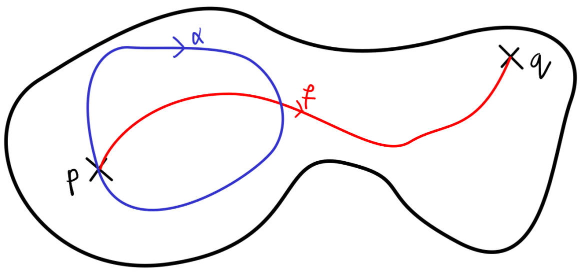


Figure 11: The loop α based at p becomes a loop $(\bar{f} \cdot \alpha) \cdot f$ based at q.[14]

Remark 3.7. Observe that we defined $\hat{f}$ (and, analogously, $\hat{\bar{f}}$) without specifying the bracketing on $\bar{f} \cdot \alpha \cdot f$ (and, analogously, $f \cdot \alpha \cdot \bar{f}$). This is because, by the associativity proved in Theorem 3.3, we know that $[(\bar{f} \cdot \alpha) \cdot f] = ([\bar{f}] \cdot [\alpha]) \cdot [f] = [\bar{f}] \cdot ([\alpha] \cdot [f]) = [\bar{f} \cdot (\alpha \cdot f)]$ (and analogously for $\hat{\bar{f}}$). Hence, we can use $[\bar{f} \cdot \alpha \cdot f]$ (and, analogously, $[f \cdot \alpha \cdot \bar{f}]$) without ambiguity.

As the choice of basepoint ultimately doesn't matter for a path-connected topological space X, the notation $\pi_1(X)$ is commonly used for the group isomorphic to $\pi_1(X, p)$ for all $p \in X$ [1, p. 94].

Definition 3.8 (Simply connected)**.** [1, p. 96] Let X be a path-connected topological space. Then, X is *simply connected* if $\pi_1(X)$ is trivial.

Example 3.9 (Convex subsets of $\mathbb{R}^n$)**.** Clearly, a convex subset C of $\mathbb{R}^n$ is path-connected, so, for $p, q \in C$, we have that $\pi_1(C, p) \cong \pi_1(C, q)$. Hence, by Example 3.5, C is simply connected.

A space being simply connected intuitively means that it has no holes - any loop can be contracted down to a single point as there are no holes to get in the way.

[14] This figure is the author's own work.

3.2 Immediate consequences

Topology can be considered the study of topological spaces and continuous functions between them [10, p. 1]. In this regard, it is desirable to study continuous functions between topological spaces, and see how these interact with the fundamental group. We'll see that the relationship is very nice, with continuous functions inducing homomorphisms.

Definition 3.10 (Induced homomorphisms)**.** [1, p.94] Let X and Y be topological spaces with $p \in X$. Let $f : X \to Y$ be continuous. Define $q \coloneqq f(p)$. Then, the function $f_* : \pi_1(X, p) \to \pi_1(Y, q)$ defined by $f_*([\alpha]) = [f \circ \alpha]$ is a homomorphism *induced* by f.

Of course, we must check that this is both well-defined and actually a homomorphism.

Proposition 3.11 (Induced homomorphisms are well-defined homomorphisms)**.** *[1, p. 94] Let X and Y be topological spaces with $p \in X$. Let $f : X \to Y$ be continuous. Define $q \coloneqq f(p)$. Let $f_* : \pi_1(X, p) \to \pi_1(Y, q)$ be the homomorphism induced by f. Then, f_* is indeed a well-defined homomorphism.*

Proof. [1, p. 94] Let α be a loop in X based at p. Then, clearly, $f \circ \alpha$ is clearly a loop in Y based at q. By Proposition 2.25, if β is also a loop in X based at p and $[\alpha] = [\beta]$, then $[f \circ \alpha] = [f \circ \beta]$. Hence, f_* is well-defined. Moreover, it is simple to check that $f \circ (\alpha \cdot \beta) = (f \circ \alpha) \cdot (f \circ \beta)$, and it follows that f_* is a homomorphism. ☐

Proposition 3.12. *[1, p. 95] Let X, Y and Z be topological spaces, and suppose we have maps $f : X \to Y$ and $g : Y \to Z$. Then, $(g \circ f)_* = g_* \circ f_*$ for any choice of basepoint.*

Proof. [15] Let $p \in X$. Then, for all $[\alpha] \in \pi_1(X, p)$, we have

$$\begin{aligned}(g \circ f)_*([\alpha]) = [(g \circ f) \circ \alpha] &= [g \circ (f \circ \alpha)] = g_*([f \circ \alpha]) \\ &= g_*(f_*([\alpha])) = (g_* \circ f_*)([\alpha]).\end{aligned}$$ ☐

Remark 3.13. [11, p. 1465] Let $C^1(X, Y)$ be the set of continuous maps from X to Y [7, p. 1], and $\mathrm{Hom}(\pi_1(X), \pi_1(Y))$ be the set of homomorphisms from $\pi_1(X)$ to $\pi_1(Y)$. Then, we have a map $(\cdot)_* : C^1(X, Y) \to \mathrm{Hom}(\pi_1(X), \pi_1(Y))$. This is an example of a *pushforward map*, and Proposition 3.12 says that it's *natural*.

It is natural to ask when isomorphisms between fundamental groups can occur.

[15]This proof is the author's own work.

Corollary 3.14 (Homeomorphic spaces have isomorphic fundamental groups)**.** *[1, p. 95] Let X and Y be homeomorphic topological spaces. Then, $\pi_1(X) \cong \pi_1(Y)$.*

Proof. [1, p. 95] Let $h : X \to Y$ be a homeomorphism. Then, applying Proposition 3.12 to the compositions $h \circ h^{-1} = \mathrm{id}_X$ and $h^{-1} \circ h = \mathrm{id}_Y$ shows us that h_* has an inverse h_*^{-1}, so h_* is an isomorphism and $\pi_1(X) \cong \pi_1(Y)$. □

This is hugely important as it shows that the fundamental group is a *topological property*—that is, it's preserved under homomorphism.

Example 3.15. Consider the disc of radius $r > 0$ in $\mathbb{R}^2$, $D_r := \{x \in \mathbb{R}^2 \mid \|x\| \leq r\}$. Then, for all $p, q > 0$, we have that D_p is homeomorphic to D_q by $f(x) = \frac{q}{p}x$. Sure enough, as D_r is clearly convex for all $r > 0$, by Example 3.5, $\pi_1(D_p) \cong \pi_1(D_q) \cong \{1\}$.

Note that we do not have the reverse implication of Corollary 3.14, as per the following example.

Example 3.16 (Indiscrete spaces)**.** [16] Let X be a non-empty set endowed with the indiscrete topology, so the only open sets in X are X and $\emptyset$. We will show that $\pi_1(X) \cong \{1\}$.

First, let Y be a topological space, and consider any function $f : Y \to X$. Then, $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(X) = Y$, both of which are open. Hence, f is continuous.

Moreover, X is path-connected. To see this, consider any $u, v \in X$. Then, define $f : [0,1] \to X$ by $f(s) = u$ for $s \in [0,1)$ and $f(1) = v$. This is continuous, as shown above, so X is path-connected.

Now, fix a point $x_0 \in X$ and consider loops α and β in X with basepoint x_0. Let $F : [0,1]^2 \to X$ be the function given by $F(s,t) = \alpha(s)$ for $t \in [0,1)$ and $F(s,1) = \beta(s)$. This is continuous, as already shown, and, for all $s \in [0,1]$, we have $F(s,0) = \alpha(s)$ and $F(s,1) = \beta(s)$. Moreover, for all $t \in [0,1]$, $F(0,t) = F(1,t) = x_0$. Hence, $\alpha \underset{F}{\simeq} \beta \operatorname{rel} \{0,1\}$, so $[\alpha] = [\beta]$. Hence, $\pi_1(X, x_0) \cong \{1\}$. As X is path-connected, we can therefore write that $\pi_1(X) \cong \{1\}$.

We therefore have that $\pi_1(X) \cong \pi_1(D_r)$ for all $r > 0$. However, X is not homeomorphic to D_r for any $r > 0$ as X only has two open subsets while D_r has infinitely many.

This highlights the fact the fundamental group is not a perfect abstraction of a space, and it only encapsulates some properties of them (namely, one-dimensional holes).

[16] This example is adapted from this author's solution to problem 12 of Armstrong §5 [1, p. 95].

So far, we've only looked at spaces with trivial fundamental groups, and they've all been relatively simple to compute. In the next section, we will briefly look at some more involved computations and see our first non-trivial fundamental groups.

4 Computations

In this section, we perform some more involved computations of fundamental groups.

4.1 Contractible spaces

A space is contractible if it can be deformed down to a single point.

Definition 4.1. [11, p. 730] Let X be a topological space. Then, X is *contractible* if, for some $p \in X$, there exists a map $f : X \to X$ such that $f(x) = p$ for all $x \in X$ and f is homotopic to the identity map, $\mathrm{id}_X : X \to X$.

Then, for any contractible space, we can use the map f and its homotopy to id_X to contract loops down to a single point.

Theorem 4.2 (Contractible spaces are simply connected)**.** *[11, p. 1464] Let X be a contractible topological space. Then, X is simply connected.*

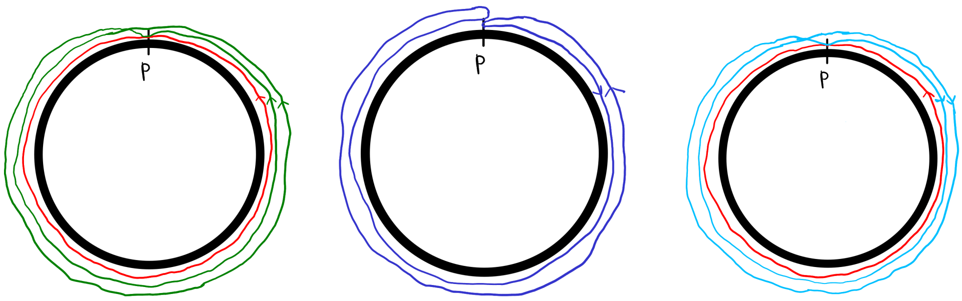
Proof. [17] Let $p \in X$ be the point guaranteed to exist by Definition 4.1, and let $f : X \to X$ be the function that maps all $x \in X$ to p. Let F_0 be a homotopy between f and id_X, so $f \underset{F_0}{\simeq} \mathrm{id}_X$. We want a homotopy relative to $\{p\}$, and there is no guarantee that this one is. Hence, define a new function $F(x,t) = F_0(x,t) - (F_0(p,t) - p)$. As F_0 is continuous, F is clearly continuous in both variables. Moreover, for all $x \in X$, $F(x,0) = F_0(x,0) - (F_0(p,0) - p) = f(x) - (f(p) - p) = f(x) - (p - p) = f(x)$, and $F(x,1) = F_0(x,1) - (F_0(p,1) - p) = \mathrm{id}_X(x) - (\mathrm{id}_X(p) - p) = \mathrm{id}_X(x)$. Moreover, for all $t \in [0,1]$, $F(p,t) = F_0(p,t) - (F_0(p,t) - p) = p$. Hence, $f \underset{F}{\simeq} \mathrm{id}_X \operatorname{rel} \{p\}$.

Now, let α be a loop in X with basepoint p. Then, $\mathrm{id}_X \circ \alpha = \alpha$, and $f \circ \alpha = \varepsilon_p$. Then, by Proposition 2.25, we have $f \circ \alpha \underset{G}{\simeq} \mathrm{id}_X \circ \alpha \operatorname{rel} \alpha^{-1}(p)$, where $G(s,t) = F(\alpha(s),t)$, so $\varepsilon_p \underset{G}{\simeq} \alpha \operatorname{rel} \alpha^{-1}(p)$. As $\{0,1\} \subseteq \alpha^{-1}(p)$, we have that $\varepsilon_p \underset{G}{\simeq} \alpha \operatorname{rel} \{0,1\}$. Hence, $[\alpha] = [\varepsilon_p]$. As this holds for all loops α in X with basepoint p, we have that all elements of $\pi_1(X,p)$ are the identity. Hence, $\pi_1(X) \cong \pi_1(X,p) \cong \{1\}$, so X is simply connected. □

[17] This proof is the author's own work.

4.2 The circle

Consider loops going around the unit circle. In particular, consider the number of times these loops “go around” the circle. Say that some loop goes around n times, and another goes around m times, with $n, m \in \mathbb{Z}$, where positive integers represent anticlockwise traversals, and negative integers represent clockwise traversals. Then, it should be clear that the product of these two paths will go around the circle $n+m$ times. This idea is illustrated in Figure 12. This should suggest that the fundamental group of the circle is isomorphic to the integers under addition, which is indeed the case.



(a) The red loop goes around 1 time, while the green loop goes around 2 times. Their product (in either order) goes around 3 times.

(b) The dark blue loop goes around 0 times.

(c) The red loop goes around 1 time, while the light blue loop goes around -2 times. Their product (in either order) goes around -1 times.

Figure 12: Examples of loops in S^1. For clarity, loops have been drawn outside of the circle, because their radii are all a constant 1 so only the angle swept by the loops matters.[18]

Theorem 4.3. *[9, p. 54] We have that $\pi_1(S^1) \cong \mathbb{Z}$.*

Proving this requires a somewhat surprising amount of work, including several lemmas. The idea is to formalise the idea of the “going around” the circle a certain number of times with a concept called the *degree* of a loop. The proof as given by Spanier [9, pp. 53-55] can be found in Appendix A.1.

The fundamental group of the circle has a few interesting applications. One of these, a proof of the Brouwer Fixed Point Theorem in dimension 2 [4, p. 31], is given below.

Theorem 4.4 (Brouwer fixed point theorem in dimension 2)**.** *[4, p. 31] Let $D = \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$ be the unit disc in $\mathbb{R}^2$, and let $h : D \to D$ be continuous. Then, h has a fixed point. That is, there exists $x \in D$ with $h(x) = x$.*

[18] These figures are the author’s own work.

Proof. [4, p. 32] Proceed by contradiction and suppose that $h(x) \neq x$ for all $x \in D$. Define a map $r : D \to S^1$ such that $r(x)$ is the point in S^1 that the ray in $\mathbb{R}^2$ starting at $h(x)$ and passing through x hits, as in Figure 13. It should then be clear that r is well-defined for all $x \in D$, as $h(x)$ and x are distinct and any two points in $\mathbb{R}^2$ define a line, and, possibly excluding $h(x)$ itself, the ray starting at $h(x)$ and passing through x will only intersect S^1 once. Moreover, r is continuous. To see this, note that it is possible, albeit very tedious, to write r as the sums and products of continuous functions.

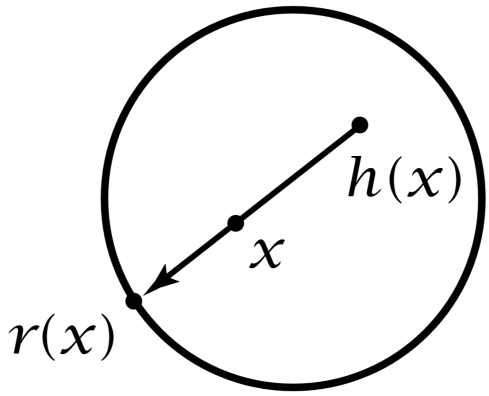


Figure 13: Computing $r(x)$ by extending the line segment from $h(x)$ to x.[19]

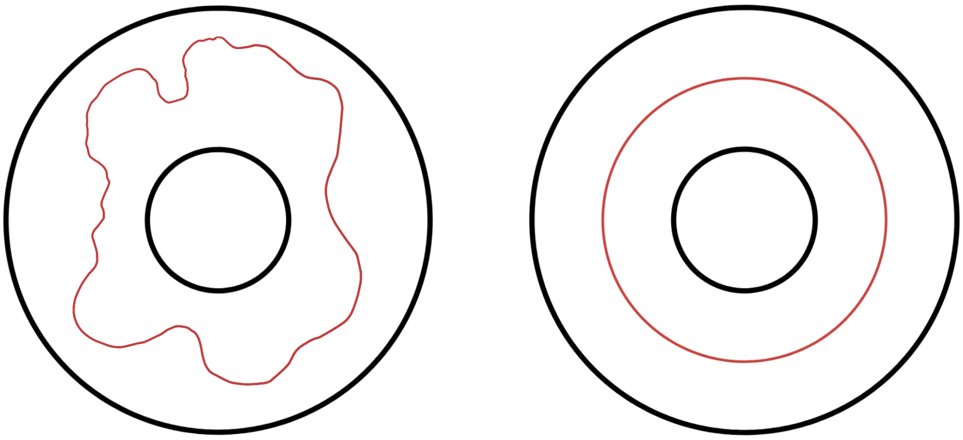
Now, let $p \in S^1$ and let α be a loop in S^1 with basepoint p. Then, considering α as a loop in D, we have $\alpha \underset{F}{\simeq} \varepsilon_p$ for some $F : [0,1]^2 \to D$ by Example 3.5. Then, $r \circ F$ is a homotopy in S^1 from $r \circ \alpha = \alpha$ to $r \circ \varepsilon_p = \varepsilon_p$, with both equalities coming from the fact that $r(x) = x$ for $x \in S^1$, so $[\alpha] = [\varepsilon_p]$. This contradicts the fact that $\pi_1(S^1, p) \cong \pi_1(S^1) \cong \mathbb{Z} \ncong \{1\}$, proving the statement. $\square$

4.3 The annulus

Let $A = \{x \in \mathbb{R}^2 \mid 1/2 \leq \|x\| \leq 3/2\}$ be an annulus in $\mathbb{R}^2$. Intuitively, A should have the same fundamental group as S^1: by considering the number of times that loops go around the hole of the annulus, we can form an isomorphism between $\pi_1(A)$ and $\mathbb{Z}$. We can prove this by going via $\pi_1(S^1)$, and this is done by "normalising" the radii of loops in A to 1, making them loops in S^1 (see Figure 14).

Theorem 4.5. [21] *We have that* $\pi_1(A) \cong \mathbb{Z}$.

[19]This figure is taken from Hatcher [4, p. 32].
[20]These figures are the author's own work.
[21]This theorem and its proof is the author's own work, based upon ideas from Armstrong §5.4.



(a) A loop in A.

(b) The result of "normalising" this loop to have a radius of 1, making it a loop in S^1.

Figure 14: "Normalising" a loop in A.[20]

Proof. Fix a basepoint $p \in A$. It will be convenient to work in polar coordinates and write $p = (r_p, \theta_p)$ for some $r_p \in [1, 2]$ and $\theta_p \in [0, 2\pi)$. Let $p' := (1, \theta_p)$ be a basepoint in S^1. Let $[\cdot]_A$ denote homotopy classes in A, and $[\cdot]_{S^1}$ denote homotopy classes in S^1. Now, define a function $f : \pi_1(A, p) \to \pi_1(S^1, p')$ by $f([\alpha]_A) = [\alpha']_{S^1}$, where, writing $\alpha(s) = (r_\alpha(s), \theta_\alpha(s))$, α' is given by $\alpha'(s) = (1, \theta_\alpha(s))$. Furthermore, define a function $g : \pi_1(S^1, p') \to \pi_1(A, p)$ by $g([\beta]_{S^1}) = [\beta']_A$, where, writing $\beta(s) = (1, \theta_\beta(s))$, β' is given by $\beta'(s) = (r_p, \theta_\beta(s))$. Now, for $[\alpha]_A \in \pi_1(A, p)$ where we write $\alpha(s) = (r_\alpha(s), \theta_\alpha(s))$, observe that

$$\begin{aligned}
(g \circ f)([\alpha]_A) &= g(f([\alpha]_A)) \\
&= g([\alpha']_{S^1}), \text{ where } \alpha'(s) = (1, \theta_\alpha(s)), \\
&= [\alpha'']_A, \text{ where } \alpha''(s) = (r_p, \theta_\alpha(s)), \\
&= [\alpha]_A, \text{ as } \alpha'' \underset{F}{\simeq} \alpha \operatorname{rel} \{0, 1\}, \text{ where} \\
&\qquad\qquad F(s, t) = ((1 - t)r_p + t r_\alpha(s), \theta_\alpha(s)),
\end{aligned}$$

and, for $[\beta]_{S^1} \in \pi_1(S^1, p')$ where we write $\beta(s) = (1, \theta_\beta(s))$, observe that

$$\begin{aligned}
(f \circ g)([\beta]_{S^1}) &= f(g([\beta]_{S^1})) \\
&= f([\beta']_A), \text{ where } \beta'(s) = (r_p, \theta_\beta(s)), \\
&= [\beta'']_{S^1}, \text{ where } \beta''(s) = (1, \theta_\beta(s)), \\
&= [\beta]_{S^1}.
\end{aligned}$$

Hence, g is an inverse for f so f is a bijection. Moreover, it can be shown via a basic calculation that, for $[\alpha]_A, [\beta]_A \in \pi_1(A, p)$, we have $f([\alpha]_A \cdot [\beta]_A) =$

$f([\alpha]_A) \cdot f([\beta]_A)$, so f is a homomorphism. Hence, f is an isomorphism, so $\pi_1(A) \cong \pi_1(A, p) \cong \pi_1(S^1, p') \cong \pi_1(S^1) \cong \mathbb{Z}$. □

5 Conclusion

Recall from Section 1 that our aim is to show that a disc is not homeomorphic to an annulus. Using Corollary 3.14, which says that the fundamental group of a space is preserved under homeomorphisms, this is now easy! We found in Example 3.15 that $\pi_1(D_r) \cong \{1\}$ for all $r > 0$, where D_r is the disc of radius r. Moreover, from Theorem 4.5, we know that, if $A = \{x \in \mathbb{R}^2 \mid 1/2 \leq \|x\| \leq 3/2\}$ is an annulus with inner radius $1/2$ and outer radius $3/2$, that $\pi_1(A) \cong \mathbb{Z}$, and this argument can easily be extended to annuli of other radii. Hence, a disc and an annulus are never homeomorphic.

This idea can easily be used on many other pairs of topological spaces in order to show that they are not homeomorphic. Theorems such as the Seifert–Van Kampen Theorem allow for the systematic computation of the fundamental groups of many more topological spaces by breaking them down into simpler spaces whose fundamental groups are already understood [2, p. 156].

Finally, we will comment briefly on the presence of the subscript 1 in the notation for the fundamental group. This 1 indicates that the fundamental group is the first entry in a sequence known as the homotopy groups. While the fundamental group deals with equivalence classes of functions with domain $[0, 1]$, the nth homotopy group instead uses functions with domain $[0, 1]^n$ [4, p. 26], and can be thought of as capturing higher-dimensional holes. For example, $\pi_n(S^n) \cong \mathbb{Z}$ [4, p. 97], where the void within the n-dimensional sphere can be thought of as an n-dimensional hole. Use of these higher homotopy groups allows for more fidelity in differentiating spaces at the cost of being harder to compute.

A Appendix: omitted proofs

This appendix contains several proofs omitted from the main essay.

Proof of Proposition 2.23. [1, p. 90] We must show that homotopy is reflexive, symmetric and transitive. For reflexiveness, observe that, for all maps $f : X \to Y$, we clearly have $f \underset{F}{\simeq} f$, where $F(x, t) = f(x)$. For symmetry, observe that, for all maps $f, g : X \to Y$ with $f \underset{F}{\simeq} g$, we have $g \underset{G}{\simeq} f$ where $G(x, t) = F(x, 1 - t)$. Finally, for transitivity, observe that, for all maps $f, g, h : X \to Y$ with $f \underset{F}{\simeq} g$ and $g \underset{G}{\simeq} h$, we have $f \underset{H}{\simeq} h$ where

$$H(x, t) = \begin{cases} F(x, 2t), & 0 \leq t \leq \frac{1}{2}, \\ G(x, 2t - 1), & \frac{1}{2} \leq t \leq 1, \end{cases}$$

which is continuous by Lemma 2.8 and clearly satisfies the other conditions for a homotopy. $\square$

Proof of Proposition 2.25. For the first statement, first note that, for all $x \in X$, $(h \circ F)(x, 0) = h(F(x, 0)) = h(f(x)) = (h \circ f)(x)$ and $(h \circ F)(x, 1) = h(F(x, 1)) = h(g(x)) = (h \circ g)(x)$. Moreover, if $x \in A$, then, for all $t, t' \in [0, 1]$, we have $(h \circ F)(x, t) = h(F(x, t)) = h(F(x, t')) = (h \circ F)(x, t')$. Finally, $h \circ F$ is the composition of two continuous functions so is continuos itself.

For the second statement, first note that, for all $x \in X$, $F(x, 0) = G(f(x), 0) = g(f(x)) = (g \circ f)(x)$ and $F(x, 1) = G(f(x), 1) = h(f(x)) = (h \circ f)(x)$. Moreover, if $x \in f^{-1}(B)$, then $f(x) \in B$ and so, for all $t, t' \in [0, 1]$, we have $F(x, t) = G(f(x), t) = G(f(x), t') = F(x, t')$. Finally, F can be written as the composition of $(x, t) \mapsto (f(x), t)$ and G, which are both continuous, so is continuous itself. $\square$

Proof of Proposition 3.2. [4, p.26] Note that, for all $s \in [0, 1]$, we have

$$\begin{aligned}
H(s, 0) &= \begin{cases} F(2s, 0), & 0 \leq s \leq \frac{1}{2}, \\ G(2s - 1, 0), & \frac{1}{2} \leq s \leq 1, \end{cases} \\
&= \begin{cases} f_0(2s), & 0 \leq s \leq \frac{1}{2}, \\ g_0(2s - 1), & \frac{1}{2} \leq s \leq 1, \end{cases} \\
&= (f_0 \cdot g_0)(s),
\end{aligned}$$

and

$$\begin{aligned}
H(s, 1) &= \begin{cases} F(2s, 1), & 0 \leq s \leq \frac{1}{2}, \\ G(2s - 1, 1), & \frac{1}{2} \leq s \leq 1, \end{cases} \\
&= \begin{cases} f_1(2s), & 0 \leq s \leq \frac{1}{2}, \\ g_1(2s - 1), & \frac{1}{2} \leq s \leq 1, \end{cases} \\
&= (f_1 \cdot g_1)(s).
\end{aligned}$$

Moreover, for all $t, t' \in [0, 1]$, we have that $H(0, t) = F(0, t) = F(0, t') = H(0, t')$ and, similarly, $H(1, t) = G(1, t) = G(1, t') = H(1, t')$. Finally, H is continuous by Lemma 2.8. $\square$

Proof of Theorem 3.3. [1, pp. 92–93] First, note that the product of the homotopy classes of loops with a fixed basepoint is a binary operation, as this follows from the equivalent fact for products of loops themselves. Hence, closure is satisfied. It remains to show associativity, the existence of an identity, and the existence of inverses.

To show associativity, first let α, β and γ be loops based at p. Hence,

$$((\alpha \cdot \beta) \cdot \gamma)(s) = \begin{cases} (\alpha \cdot \beta)(2s), & 0 \le s \le \frac{1}{2}, \\ \gamma(2s-1), & \frac{1}{2} \le s \le 1, \end{cases}$$
$$= \begin{cases} \alpha(4s), & 0 \le s \le \frac{1}{4}, \\ \beta(4s-1), & \frac{1}{4} \le s \le \frac{1}{2}, \\ \gamma(2s-1), & \frac{1}{2} \le s \le 1, \end{cases}$$

and

$$(\alpha \cdot (\beta \cdot \gamma))(s) = \begin{cases} \alpha(2s), & 0 \le s \le \frac{1}{2}, \\ (\beta \cdot \gamma)(2s-1), & \frac{1}{2} \le s \le 1, \end{cases}$$
$$= \begin{cases} \alpha(2s), & 0 \le s \le \frac{1}{2}, \\ \beta(4s-2), & \frac{1}{2} \le s \le \frac{3}{4}, \\ \gamma(4s-3), & \frac{3}{4} \le s \le 1. \end{cases}$$

Therefore, we have that $(\alpha \cdot \beta) \cdot \gamma = (\alpha \cdot (\beta \cdot \gamma)) \circ f$, where $f : [0,1] \to [0,1]$ is given by

$$f(s) = \begin{cases} 2s, & 0 \le s \le \frac{1}{4}, \\ s + \frac{1}{4}, & \frac{1}{4} \le s \le \frac{1}{2}, \\ \frac{s+1}{2}, & \frac{1}{2} \le s \le 1. \end{cases}$$

Note that f is continuous by Lemma 2.8. Moreover, $[0,1]$ is convex so there exists a straight-line homotopy (see Example 2.14) from f to $\mathrm{id}_{[0,1]}$, the identity map for $[0,1]$, relative to $\{0,1\}$. Hence, by Proposition 2.25, we have that $(\alpha \cdot (\beta \cdot \gamma)) \circ f \simeq (\alpha \cdot (\beta \cdot \gamma)) \circ f \, \mathrm{id}_{[0,1]} \operatorname{rel} \{0,1\}$. Hence, as $(\alpha \cdot \beta) \cdot \gamma = (\alpha \cdot (\beta \cdot \gamma)) \circ f$ and $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot (\beta \cdot \gamma)) \circ \mathrm{id}_{[0,1]}$, we have that $(\alpha \cdot \beta) \cdot \gamma \simeq \alpha \cdot (\beta \cdot \gamma) \operatorname{rel} \{0,1\}$, so $[(\alpha \cdot \beta) \cdot \gamma] = [\alpha \cdot (\beta \cdot \gamma)]$. Hence, $([\alpha] \cdot [\beta]) \cdot [\gamma] = [\alpha] \cdot ([\beta] \cdot [\gamma])$.

The identity element is $[\varepsilon_p]$, the homotopy class of the constant path at p (see Definition 2.5). Let α be a loop based at p. Observe that $\varepsilon_p \cdot \alpha = \alpha \circ f$, where $f : [0,1] \to [0,1]$ is given by

$$f(s) = \begin{cases} 0, & 0 \le s \le \frac{1}{2}, \\ 2s-1, & \frac{1}{2} \le s \le 1. \end{cases}$$

Therefore, f is continuous by Lemma 2.8. Since $f(0) = f(1) = 0$ and $[0,1]$ is convex, we know that $f \simeq \mathrm{id}_{[0,1]} \operatorname{rel} \{0,1\}$ via a straight-line homotopy (Example 2.14). Hence, by Proposition 2.25, $\alpha \circ f \simeq \alpha \circ \mathrm{id}_{[0,1]} \operatorname{rel} \{0,1\}$, so, as $\varepsilon_p \cdot \alpha = \alpha \circ f$ and $\alpha = \alpha \circ \mathrm{id}_{[0,1]}$, we have that $\varepsilon_p \cdot \alpha \simeq \alpha \operatorname{rel} \{0,1\}$, so $[\varepsilon_p \cdot \alpha] = [\alpha]$, so $[\varepsilon_p] \cdot [\alpha] = [\alpha]$. The fact that $[\alpha] \cdot [\varepsilon_p] = [\alpha]$ follows in a very similar way.

Finally, for inverses, let α be a loop based at p. Its inverse is the homotopy class $[\bar{\alpha}]$ of the loop $\bar{\alpha}$ defined by $\bar{\alpha}(s) = \alpha(1-s)$ [4, p. 27]. This inverse is well-defined since, if β is also a loop based at p such that $\alpha \underset{F}{\simeq} \beta \operatorname{rel} \{0,1\}$, then $\bar{\alpha} \underset{G}{\simeq} \bar{\beta} \operatorname{rel} \{0,1\}$ where $G(s,t) = F(1-s,t)$. Now, note that $\alpha \cdot \bar{\alpha} = \alpha \circ f$, where $f : [0,1] \to [0,1]$ is given by

$$f(s) = \begin{cases} 2s, & 0 \le s \le \frac{1}{2}, \\ 2-2s, & \frac{1}{2} \le s \le 1. \end{cases}$$

Therefore, f is continuous by Lemma 2.8. Since $f(0) = f(1) = 0$ and $[0,1]$ is convex, we know that $f \simeq \varepsilon_0 \operatorname{rel} \{0,1\}$ via a straight-line homotopy (Example 2.14). Hence, by Proposition 2.25, $\alpha \circ f \simeq \alpha \circ \varepsilon_p \operatorname{rel} \{0,1\}$. Therefore, as $\alpha \cdot \bar{\alpha} = \alpha \circ f$ and $\varepsilon_p = \alpha \circ \varepsilon_0$, we have that $\alpha \cdot \bar{\alpha} \simeq \varepsilon_p \operatorname{rel} \{0,1\}$, so $[\alpha \cdot \bar{\alpha}] = [\varepsilon_p]$, so $[\alpha] \cdot [\bar{\alpha}] = [\varepsilon_p]$. The fact that $[\bar{\alpha}] \cdot [\alpha] = [\varepsilon_p]$ follows in a very similar way. $\square$

A.1 Proof of the fundamental group of the circle

In this subsection, we prove Theorem 4.3.

Throughout, we will identify S^1 with the unit circle in the complex plane. That is, $S^1 = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$ [9, p. 53]. Formally, this is via the bijection $(\cos\theta, \sin\theta) \mapsto e^{i\theta}$. This identification makes the proof of Theorem 4.3 much easier and neater. Moreover, this computation is easier if we pick a basepoint for our loops. We will pick the point $1 \in S^1$. As per Theorem 3.6, this choice doesn't matter.

The key in defining this degree concept is to consider *lifts*: consider a loop $\alpha : [0,1] \to S^1$ based at 1. Its lift is a function $\tilde{\alpha} : [0,1] \to \mathbb{R}$ such that, for the correct function g, we have that $\tilde{\alpha}(0) = t_0$ and $g(t_0) = \alpha(0)$ for some $t_0 \in \mathbb{R}$, and $\alpha = g \circ \tilde{\alpha}$ [4, p. 29]. It turns out that the correct function g for our purpose is as follows.

Definition A.1 (Exponential map)**.** [9, p. 53] The *exponential map* is the function $\operatorname{ex} : \mathbb{R} \to S^1$ defined by $\operatorname{ex}(t) = e^{2\pi i t}$.

This should make intuitive sense: it is well known that the complex exponential captures rotation nicely, and embedding the constant 2π into the function means that a simple loop going around the circle once, $s \mapsto e^{2\pi i s} : [0,1] \to S^1$, lifts to the simple function $s \mapsto s : [0,1] \to \mathbb{R}$.

Some basic facts about ex can be immediately derived from the properties of the exponential.

Proposition A.2. *[9, p. 53]*

(i) The exponential map is continuous;

(ii) for all $t_1, t_2 \in \mathbb{R}$, $\operatorname{ex}(t_1 + t_2) = \operatorname{ex}(t_1)\operatorname{ex}(t_2)$*;*

(iii) for all $t_1, t_2 \in \mathbb{R}$*,* $\operatorname{ex}(t_1 - t_2) = \frac{\operatorname{ex}(t_1)}{\operatorname{ex}(t_2)}$.

We'd now like to show that all loops $[0,1] \to S^1$ have a lift. However, we will also need this fact for functions $[0,1]^2 \to S^1$. It turns out that it is no harder to show this for all functions whose domains are *starlike* subsets of $\mathbb{R}^n$.

Definition A.3. [9, p. 53] Let X be a subset of $\mathbb{R}^n$ and $x_0 \in X$. The subset X is called *starlike* from x_0 if, whenever $x \in X$, the closed line segment from x_0 to x lies in X.

The lemma holds for subsets starlike from any $x_0 \in X$, but we will prove it only from 0. It is easy to generalise the proof to more general points by incorporating a translation.

We also impose the condition that X is compact for reasons that will become apparent in the proof.

Lemma A.4. *[9, p. 53] Let* X *be a compact subset of* $\mathbb{R}^n$ *with* $0 \in X$ *and* X *starlike from* 0*. Let* $f : X \to S^1$ *be a map and* $t_0 \in \mathbb{R}$ *such that* $\operatorname{ex}(t_0) = f(0)$*. Then, there exists a map* $\tilde{f} : X \to \mathbb{R}$ *such that* $\tilde{f}(0) = t_0$ *and, for all* $x \in X$*,* $\operatorname{ex}(\tilde{f}(x)) = f(x)$*.*

In a simple world, we'd consider a continuous inverse of ex, which we'll call lg and leave ambiguously defined temporarily, and let $\tilde{f} = \lg(f(x))$. However, as ex is not injective, such a lg would not be well-defined. Therefore, in order to ensure that lg is well-defined, we should restrict the domain of ex, say to $\left(-\frac{1}{2}, \frac{1}{2}\right]$. However, we still would have a problem, as then $\lg : S^1 \to \left(-\frac{1}{2}, \frac{1}{2}\right]$ would not be continuous at $e^{\pi i} = -1$, as $\lg(x)$ would be close to $\frac{1}{2}$ when x is "just above" $e^{\pi i}$, but close to $-\frac{1}{2}$ when x is "just below" $e^{\pi i}$. Hence, to solve this issue, we must remove $\frac{1}{2}$ from the restricted domain of ex, giving us the following definition of its inverse function, lg.

Definition A.5 (The inverse of ex restricted to $\left(-\frac{1}{2}, \frac{1}{2}\right)$)**.** [9, p. 53] The *inverse function* to $\operatorname{ex}|_{\left(-\frac{1}{2}, \frac{1}{2}\right)}$ is denoted $\lg : S^1 \setminus \{e^{\pi i}\} \to \left(-\frac{1}{2}, \frac{1}{2}\right)$.

As defined here, lg is clearly continuous. However, this definition is problematic as the domain of lg excludes $e^{\pi i}$, and it might be the case that, for some $x \in X$, $f(x) = e^{\pi i}$. To get around this, we consider quotients $\frac{f(x)}{f(x')}$ and use those to build up f and, hence, a lift of f using lg, and use the uniform continuity of f implied by the compactness of X to control these quotients and ensure that they are not equal to $e^{\pi i}$.

Proof of Lemma A.4. [9, pp. 53–54] As X is compact, we know that f must be uniformly continuous. In particularly, this tells us that there exists an $\varepsilon > 0$ such that, for all $x, x' \in X$ with $\|x - x'\| < \varepsilon$, then $|f(x) - f(x')| < 2$, where $|\cdot|$ denotes the complex modulus. This means that, when $x, x' \in X$

and $\|x - x'\| < \varepsilon$, $f(x)$ and $f(x')$ are not antipodal (i.e. $f(x) \neq -f(x')$). In particular, this means that $\frac{f(x)}{f(x')} \neq -1 = e^{\pi i}$.

Also as X is compact, we know that X is bounded, and so there exists a positive integer $m \in \mathbb{Z}_+$ such that, for all $x \in X$, we have $\|x\| < m\varepsilon$, and so $\frac{\|x\|}{m} < \varepsilon$.

Fix an $x \in X$. As X is starlike from 0, the closed line segment from 0 to x is in X. In particular, for all $j \in \{0, 1, \dots, m-1\}$, $\frac{jx}{m} \in X$ and $\frac{(j+1)x}{m} \in X$.

As the image of f is S^1, we clearly have that

$$\left\| \frac{f((j+1)x/m)}{f(jx/m)} \right\| = \frac{\|f((j+1)x/m)\|}{\|f(jx/m)\|} = \frac{1}{1} = 1,$$

so $\frac{f((j+1)x/m)}{f(jx/m)} \in S^1$. Note that, as

$$\left\| \frac{(j+1)x}{m} - \frac{jx}{m} \right\| = \frac{\|x\|}{m} < \varepsilon,$$

we have

$$\left| f\left(\frac{(j+1)x}{m}\right) - f\left(\frac{jx}{m}\right) \right| < 2.$$

Hence, $\frac{f((j+1)x/m)}{f(jx/m)} \in S^1 \setminus \{e^{\pi i}\}$.

Now, for all $j \in \{0, 1, \dots, m-1\}$, $\frac{jx}{m} \in X$, define a function $g_j : X \to S^1 \setminus \{e^{\pi i}\}$ by

$$g_j(x) \coloneqq \frac{f((j+1)x/m)}{f(jx/m)}.$$

Then,

$$\begin{aligned} f(0)g_0(x)g_1(x)\cdots g_{m-1}(x) =& f(0)\frac{f(x/m)}{f(0x/m)}\frac{f(2x/m)}{f(x/m)}\cdots\frac{f(mx/m)}{f((m-1)x/m)} \\ =& f(0)\frac{f(x/m)}{f(0)}\frac{f(2x/m)}{f(x/m)}\cdots\frac{f(x)}{f((m-1)x/m)} \\ =& f(x). \end{aligned}$$

Now, define $\tilde{f} : X \to \mathbb{R}$ by

$$\tilde{f}(x) \coloneqq t_0 + \lg(g_0(x)) + \lg(g_1(x)) + \cdots + \lg(g_{m-1}(x)).$$

This is a sum of continuous functions and hence is continuous itself. Moreover,

$$\begin{aligned} \tilde{f}(0) &= t_0 + \lg(g_0(0)) + \lg(g_1(0)) + \cdots + \lg(g_{m-1}(0)) \\ &= t_0 + \lg(1) + \lg(1) + \cdots + \lg(1) \\ &= t_0 + 0 + 0 + \cdots + 0, \text{ as } \mathrm{ex}(0) = 1, \text{ so } \lg(1) = 0, \\ &= t_0. \end{aligned}$$

Finally,

$$\begin{aligned}
\mathrm{ex}(\tilde{f}(x)) &= \mathrm{ex}(t_0 + \lg(g_0(x)) + \lg(g_1(x)) + \cdots + \lg(g_{m-1}(x))) \\
&= \mathrm{ex}(t_0)\,\mathrm{ex}(\lg(g_0(x)))\,\mathrm{ex}(\lg(g_1(x)))\cdots\mathrm{ex}(\lg(g_{m-1}(x))) \\
&= f(0)g_0(x)\cdots g_{m-1}(x) \\
&= f(x).
\end{aligned}$$

□

Remark A.6. The proof of Lemma A.4 given here is constructive and provides a method to compute the lift of a loop. Consider the simple loop $f : [0,1] \to S^1$ given by $f(s) = e^{2\pi i s}$. It is clear that this loop just goes around the circle once anticlockwise. By inspection, it has a lift $\tilde{f}(s) = s$. Let's check if the method of the proof above gives the same result.

We can see that $\varepsilon = \frac{1}{2}$ works for the uniform continuity. Clearly, for all $s \in [0,1]$, we have $\|s\| = s < 2$, so we can take $n = 4$ to get that $\frac{\|s\|}{n} = \frac{s}{n} < \varepsilon$. Hence, let

$$\begin{aligned}
g_0(s) &= \frac{f(s/4)}{f(0)} = f(s/4), & g_1(s) &= \frac{f(2s/4)}{f(s/4)} = \frac{f(s/2)}{f(s/4)}, \\
g_2(s) &= \frac{f(3s/4)}{f(2s/4)} = \frac{f(3s/4)}{f(s/2)}, & g_3(s) &= \frac{f(4s/4)}{f(3s/4)} = \frac{f(s)}{f(3s/4)}.
\end{aligned}$$

Then, let $\tilde{f} : [0,1] \to \mathbb{R}$ be given by $\tilde{f}(s) = 0 + \lg(g_0(s)) + \lg(g_1(s)) + \lg(g_2(s)) + \lg(g_3(s))$. Hence,

$$\begin{aligned}
\tilde{f}(s) &= \lg(f(s/4)) + \lg\left(\frac{f(s/2)}{f(s/4)}\right) + \lg\left(\frac{f(3s/4)}{f(s/2)}\right) + \lg\left(\frac{f(s)}{f(3s/4)}\right) \\
&= \lg(e^{\pi i s/2}) + \lg\left(\frac{e^{\pi i s}}{e^{\pi i s/2}}\right) + \lg\left(\frac{e^{3\pi i s/2}}{e^{\pi i s}}\right) + \lg\left(\frac{e^{2\pi i s}}{e^{3\pi i s/2}}\right) \\
&= \lg(e^{\pi i s/2}) + \lg(e^{\pi i s/2}) + \lg(e^{\pi i s/2}) + \lg(e^{\pi i s/2}) \\
&= \lg(\mathrm{ex}(s/4)) + \lg(\mathrm{ex}(s/4)) + \lg(\mathrm{ex}(s/4)) + \lg(\mathrm{ex}(s/4)) \\
&= \frac{s}{4} + \frac{s}{4} + \frac{s}{4} + \frac{s}{4} = s,
\end{aligned}$$

as expected.

We could also do this procedure with, say, the bound $\|s\| = s < 3$ and, hence, $n = 6$. It would be ideal if this also led to the lift $\tilde{f}(s) = s$. Hence, we would like to show the uniqueness of lifts, which is the content of the next lemma.

Lemma A.7. *[9, p. 54] Let X be a connected subset of $\mathbb{R}^n$, and consider maps $\tilde{f}, \tilde{g} : X \to \mathbb{R}$ where $\mathrm{ex} \circ \tilde{f} = \mathrm{ex} \circ \tilde{g}$, and, for some $x_0 \in X$, $\tilde{f}(x_0) = \tilde{g}(x_0)$. Then, $\tilde{f} = \tilde{g}$.*

Proof. [9, p. 54] Let $h \coloneqq \tilde{f} - \tilde{g}$. Note that h is clearly continuous. Also,

$$\begin{aligned}
\mathrm{ex}(h(x)) &= \mathrm{ex}(\tilde{f}(x) - \tilde{g}(x)) \\
&= \frac{\mathrm{ex}(\tilde{f}(x))}{\mathrm{ex}(g'(x))} \\
&= \frac{(\mathrm{ex} \circ \tilde{f})(x)}{(\mathrm{ex} \circ \tilde{g})(x)} \\
&= 1, \text{ as } \mathrm{ex} \circ \tilde{f} = \mathrm{ex} \circ \tilde{g}.
\end{aligned}$$

Hence, h must be a continuous map from X to $\mathbb{Z}$. Then, h must be constant as X is connected so $h(X)$ is connected, and the only non-empty connected subsets of $\mathbb{Z}$ are singletons [6]. Moreover, as $\tilde{f}(x_0) = \tilde{g}(x_0)$, $h(x_0) = \tilde{f}(x_0) - \tilde{g}(x_0) = 0$, so $h(x) = 0$ for all $x \in X$. Hence, $\tilde{f}(x) - \tilde{g}(x) = 0$ for all $x \in X$, so $\tilde{f} = \tilde{g}$. ☐

Consider a loop α in S^1 based at 1. As $[0,1]$ is compact and starlike from 0, and because $\alpha(0) = 1 = \mathrm{ex}(0)$, there exists a lift $\tilde{\alpha} : [0,1] \to \mathbb{R}$ such that $\tilde{\alpha}(0) = 0$ and $\mathrm{ex} \circ \tilde{\alpha} = \alpha$ by Lemma A.4. Moreover, by Lemma A.7, as $[0,1]$ is connected, this lift is unique. Note also that $\tilde{\alpha}(1)$ is an integer because $\mathrm{ex}(\tilde{\alpha}(1)) = \alpha(1) = 1$. This ensures that the following definition is well-defined.

Definition A.8 (Degree of a loop in S^1)**.** [9, p. 54] Let α be a loop in S^1 based at 1. Let $\tilde{\alpha}$ be the unique function such that $\tilde{\alpha}(0) = 0$ and $\mathrm{ex} \circ \tilde{\alpha} = \alpha$. Then, the *degree* of α is the integer $\deg \alpha = \tilde{\alpha}(1)$.

Intuitively, the degree of a loop describes how many times the loop "goes around" the circle anticlockwise, with negative values indicating a clockwise traversal instead. For instance, the loop $s \mapsto e^{2\pi i s}$ has degree 1. This works because multiplication by $e^{i\theta}$ represents a rotation in the complex plane anticlockwise by θ radians. A rotation by 2π radians is a full turn, so $e^{2\pi i n}$ represents n full rotations of the circle. Hence, if a loop α based at 1 has $\deg \alpha = n$, then $\alpha(1) = (\mathrm{ex} \circ \tilde{\alpha})(1) = \mathrm{ex}(\tilde{\alpha}(1)) = \mathrm{ex}(\deg \alpha) = \mathrm{ex}(n) = e^{2\pi i n}$, so the loop ends having completed n full rotations.

Helpfully, this notion is preserved by homotopy, as the following lemma shows.

Lemma A.9. *[9, p. 54] Let α and β be loops in S^1 with basepoint 1 such that $\alpha \underset{F}{\simeq} \beta \operatorname{rel} \{0,1\}$. Then, $\deg \alpha = \deg \beta$.*

The idea behind this proof is to lift the homotopy F, and use that to relate the lifts of α and β.

Proof. [9, p. 54] We know that the homotopy between α and β is $F : [0,1]^2 \to S^1$. Note that $[0,1]^2$ is compact and starlike from 0. Hence, by Lemma A.4, there exists a function $\tilde{F} : [0,1]^2 \to \mathbb{R}$ such that $\tilde{F}(0,0) = 0$ and $\mathrm{ex} \circ \tilde{F} = F$. Since F is a homotopy relative to $\{0,1\}$, we know that $F(0,t) = F(1,t) = 1$ for all $t \in [0,1]$. Hence, we must have that $\mathrm{ex}(\tilde{F}(0,t)) = \mathrm{ex}(\tilde{F}(1,t)) = 1$, so $\tilde{F}(0,t), \tilde{F}(1,t) \in \mathbb{Z}$, for all $t \in [0,1]$. Hence, as $[0,1]$ is connected and F is continuous, we know that $\tilde{F}(0,t)$ and $\tilde{F}(1,t)$ are constant as functions of t for the same reasoning as h is in the proof of Lemma A.7 [6]. Recall that $\tilde{F}(0,0) = 0$. Therefore, $\tilde{F}(0,t) = 0$ for all $t \in [0,1]$.

Now, let $\tilde{\alpha}, \tilde{\beta} : [0,1] \to \mathbb{R}$ be defined by $\tilde{\alpha}(s) = \tilde{F}(s,0)$ and $\tilde{\beta}(s) = \tilde{F}(s,1)$. Then, we have $\tilde{\alpha}(0) = \tilde{F}(0,0) = 0$ and $\mathrm{ex} \circ \tilde{\alpha} = \alpha$ and, similarly, $\tilde{\beta}(0) = 0$ and $\mathrm{ex} \circ \tilde{\beta} = \beta$, justify the notation used for these two functions. Hence, $\deg \alpha = \tilde{\alpha}(1) = \tilde{F}(1,0)$, and $\deg \beta = \tilde{\beta}(1) = \tilde{F}(1,1)$. Hence, as $\tilde{F}(1,t)$ is constant, we know that $\tilde{F}(1,0) = \tilde{F}(1,1)$ so $\deg \alpha = \deg \beta$. ☐

This means that we can define a function $\deg : \pi_1(S^1, 1) \to \mathbb{Z}$ by $\deg[\alpha] = \deg \alpha$ as the previous Lemma A.9 ensures that this will be well-defined.

In order to complete our goal of proving Theorem 4.3, we now need to show that deg is an isomorphism from $\pi_1(S^1, 1)$ to $\mathbb{Z}$. We will first show that deg is a homomorphism, and then that it is a bijection.

Lemma A.10 (deg is a homomorphism). *[9, p. 54] The function* deg *is a homomorphism.*

Proof. [22] Let $[\alpha], [\beta] \in \pi_1(S^1, 1)$. Then, let $\tilde{\alpha}, \tilde{\beta} : [0,1] \to \mathbb{R}$ such that $\tilde{\alpha}(0) = 0$ and $\mathrm{ex} \circ \tilde{\alpha} = \alpha$, and $\tilde{\beta}(0) = 0$ and $\mathrm{ex} \circ \tilde{\beta} = \beta$.

Consider $\alpha \cdot \beta$. By Lemma A.4, there exists a function $(\widetilde{\alpha \cdot \beta}) : [0,1] \to \mathbb{R}$ such that $(\widetilde{\alpha \cdot \beta})(0) = 0$ and $\mathrm{ex} \circ (\widetilde{\alpha \cdot \beta}) = \alpha \cdot \beta$.

Consider the path $f : [0,1] \to \mathbb{R}$ given by

$$f(s) = \begin{cases} \tilde{\alpha}(2s), & 0 \leq s \leq \frac{1}{2}, \\ \tilde{\beta}(2s-1) + \tilde{\alpha}(1), & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Observe that $f(0) = \tilde{\alpha}(0) = 0$, and

$$\begin{aligned}
(\mathrm{ex} \circ f)(s) &= \begin{cases} \mathrm{ex}(\tilde{\alpha}(2s)), & 0 \leq s \leq \frac{1}{2}, \\ \mathrm{ex}(\tilde{\beta}(2s-1) + \tilde{\alpha}(1)), & \frac{1}{2} \leq s \leq 1, \end{cases} \\
&= \begin{cases} \alpha(2s), & 0 \leq s \leq \frac{1}{2}, \\ \mathrm{ex}(\tilde{\beta}(2s-1))\,\mathrm{ex}(\tilde{\alpha}(1)), & \frac{1}{2} \leq s \leq 1, \end{cases} \\
&= \begin{cases} \alpha(2s), & 0 \leq s \leq \frac{1}{2}, \\ \beta(2s-1)\alpha(1), & \frac{1}{2} \leq s \leq 1, \end{cases} \\
&= \begin{cases} \alpha(2s), & 0 \leq s \leq \frac{1}{2}, \\ \beta(2s-1), & \frac{1}{2} \leq s \leq 1, \end{cases} \\
&= (\alpha \cdot \beta)(s).
\end{aligned}$$

[22] This proof is this author's own work.

Hence, by Lemma A.7, $(\widetilde{\alpha \cdot \beta}) = f$.

Hence,

$$\begin{aligned}
\deg[\alpha \cdot \beta] &= \deg(\alpha \cdot \beta) \\
&= (\widetilde{\alpha \cdot \beta})(1) \\
&= f(1) \\
&= \tilde{\beta}(2 \cdot 1 - 1) + \tilde{\alpha}(1) \\
&= \tilde{\beta}(1) + \tilde{\alpha}(1) \\
&= \deg \alpha + \deg \beta \\
&= \deg[\alpha] + \deg[\beta],
\end{aligned}$$

so deg is a homomorphism. □

Lemma A.11 (deg is a bijection)**.** *[9, p. 55] The function* deg *is a bijection.*

Proof. [9, p. 55] To show that deg is surjective, let $n \in \mathbb{Z}$. Then, we can define a path $\tilde{\alpha}_n$ in $\mathbb{R}$ by $\tilde{\alpha}_n(s) = sn$. Note that $\tilde{\alpha}_n(0) = 0$. We can define a loop in S^1, based at 1, by $\alpha_n = \mathrm{ex} \circ \tilde{\alpha}_n$. Clearly, we then have that $\deg[\alpha_n] = \tilde{\alpha}_n(1) = n$.

To show that deg is injective, we will aim to show that $\ker(\deg) = \{[\varepsilon_1]\}$, noting that $[\varepsilon_1]$ is the identity in $\pi_1(S^1, 1)$. To do this, consider $[\alpha] \in \pi_1(S^1, 1)$ such that $\deg[\alpha] = 0$. Then, there exists (Lemma A.4) a unique (Lemma A.7) path $\tilde{\alpha}$ in $\mathbb{R}$ with $\tilde{\alpha}(0) = 0$ and $\mathrm{ex} \circ \tilde{\alpha} = \alpha$. As $\deg[\alpha] = 0$, we know that $\tilde{\alpha}(1) = 0 = \tilde{\alpha}(0)$. In other words, $\tilde{\alpha}$ is a loop. As the fundamental group of $\mathbb{R}$ is trivial, we know that $\tilde{\alpha} \simeq \varepsilon_0 \operatorname{rel} \{0, 1\}$, the constant loop at 0 in $\mathbb{R}$. Hence, by Proposition 2.25, we must have that $\mathrm{ex} \circ \tilde{\alpha} \simeq \mathrm{ex} \circ \varepsilon_0$. Observe that, clearly, $\mathrm{ex} \circ \varepsilon_0 = \varepsilon_1$. Hence, $\alpha \simeq \varepsilon_1$, so $[\alpha] = [\varepsilon_1]$. Hence, $\ker(\deg) = \{[\varepsilon_1]\}$.

Hence, deg is a bijection. □

Proof of Theorem 4.3. From Lemma A.10 and Lemma A.11, we immediately see that $\pi_1(S^1, 1) \cong \mathbb{Z}$. Hence, using Theorem 3.6, we have Theorem 4.3. □

The ideas used in this proof can be developed to work on more general covering spaces [4, pp. 30, 56].

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